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LINEAR PROGRAMMING MODELS WITH CHANGEABLE PARAMETERS — THEORETICAL ANALYSIS ON "TAKING LOSS AT THE ORDERING TIME AND MAKING PROFIT AT THE DELIVERY TIME"

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Corporate competence sets can be expanded through capital investment and be dynamically changed overtime, which can explain the phenomenon of "taking loss at the ordering time and making profit at the time of delivery". Such phenomenon has existed in practice for a long time, but there are no mathematical model that can explain it adequately. This paper utilizes multiple criteria and multiple constraint levels linear programming (MC^{2}LP) model and its extended techniques to explore the linear programming models with changeable parameters. The parameters include: unit profit, available resources, and input-output coefficients of production function. With those parameters changed with capital investment and/or time, we study how to find dynamic best solutions to make "taking loss at the ordering time and making profit at the time of delivery" feasible. For more general cases we also sketch a generalized mathematical programming model with changeable parameters and control variables.

Keywords: Competence set; time dynamics; multiple criteria decision making; multiple criteria and multiple constraint levels linear programming.

1. Introduction

Why many innovative companies, especially in high-tech industries, are willing to take orders which offer deficits (red figures) at the ordering time? Because in their mind, perhaps, based on their calculation or intuition, they could eventually make profit (black figures) at the delivery time. The time interval from ordering time to delivery time offers the precious window of opportunity for the

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companies to improve their related technology, market conditions and resource availability.

Similarly, many companies are willing to introduce new products or services which can only offer deficits (red figures) at the time of introduction because their managements believe that eventually their products or services can make profits (black figures) in the due time. The celebrated product Walkman of Sony Inc. is a notable example. It was estimated that Sony would lose $35 for each item sold at the time of first introduction (1979). With changes of parameters, including technology improvement, market conditions, and resource availability, Walkman eventually reaps big profit for Sony Inc. (For the details of Walkman transition, see Ref. 1.)

This research has been motivated by the above observation. For simplicity, the phenomenon described above will be called "*Red in-Black out*" phenomenon. We want to explain such phenomenon by using "programming models in changeable space". Especially we will use multi-criteria and multi-constraint level (MC^2) simplex method to analyze the phenomenon, and show how we can fine tune our computation so that "Red in-Black out" can indeed become a vital business strategy in competition. We will formulate the problems into MC^2-simplex models depending on that the parameter changes are caused either by purposeful investment or by predicted trend of changes. The parameters under consideration includes objective coefficients that reflect changes of market condition, resources availability that could be changed by investment or outsourcing, and productivity coefficients that could be changed by technology and production improvement.

Note that because of changes of relevant parameters, innovative companies are willing to take risk as to have "Red in-Black out" phenomenon. By adapting and/or controlling the changes of the parameters, companies can eventually reap handsome profit. In terms of habitual domain theory,[2,3] this is a proactive attitude toward changes. The vision of strategic planning is over the entire domains within which the parameters could changes. The problems can be studied by using MC^2-simplex method, which consider all possible changes of the parameters. We did not use the formats of sensitivity analysis or parameter variation method such as those studied by Bradley *et al.*,[4] Wendell,[5–7] Hiller and Lieberman,[8] and Gal,[9] because such formats basically are of local properties, not global or entire space of changes. They inherit passive, not proactive attitude, and could not study our problems fully. However, we notice that the format of sensitivity analysis and parameter variation could still provide useful information in other setting of decision making.

This paper is organized in the following order. In Sec. 2, we will briefly sketch needed notation, format and results of MC^2-simplex method to facilitate subsequent description. In Sec. 3, we will discuss the parameter changes over the objective function and resource availability through investment effort. In Sec. 4, we will explain "Red in-Black out" phenomenon using the trend of parameter changes due to time advancement. In Sec. 5, we explore the parameter changes due to technology or productivity advancement. Further comment and conclusion will be provided in Sec. 6.

2. Preliminary: MC^2-Simplex Method

A typical linear programming model has the following format.

$$\begin{aligned}
\max\ & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t.}\ & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le d_1, \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le d_2, \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le d_m, \\
& x_j \ge 0, \quad j = 1, 2, \ldots, n.
\end{aligned} \tag{1}$$

Let $x = [x_1x_2 \cdots x_n]^{\mathrm{T}}$ be the decision vector; $c = [c_1c_2 \cdots c_n]^{\mathrm{T}}$, the objective coefficient vector; $A = [a_{ij}](i = 1, \ldots, m; j = 1, \ldots, n)$, the resource consumption (or productivity) matrix; and $d = [d_1\ d_2\ \ldots\ d_m]^{\mathrm{T}}$, the resource availability vector.

Then problem (1) can be represented by matrix form, as shown in Eq. (2).

$$\begin{aligned}
\max\ & cx \\
\text{s.t.}\ & Ax \le d, \\
& x \ge 0.
\end{aligned} \tag{2}$$

For product mix optimization problem, the element of decision vector x represents the production unit; the element of vector c, the unit profit for each product; the element in matrix A, the consumption unit of different resources by different products; and the element of vector d, the available level for each type of resources.

In multiple-criteria and multiple-constraint level (MC^2) simplex method, there are multiple criteria: $C = [c^1c^2 \ldots c^q]^{\mathrm{T}}$ is a $q \times n$ matrix, where c^k, $k = 1, \ldots, q$, is a n-dimension vector representing the kth criteria; and there are r multiple constraint levels: $D = [d^1d^2 \ldots d^r]$ is a $m \times r$ matrix, where d^k, $k = 1, \ldots, r$, is the kth constraint levels for the resources. The MC^2-simplex problem thus has the following format:

$$\begin{aligned}
\max\ & Cx \\
\text{s.t.}\ & Ax \le D, \\
& x \ge 0.
\end{aligned} \tag{3}$$

In MC^2-simplex literature[10–14], x^0 is a *potential solution* to problem (3) if there is a pair of weight vectors $(\lambda,\ \sigma)$, $\lambda > 0$, $\sigma > 0$ such that x^0 solves the following problem:

$$\begin{aligned}
\max\ & \lambda Cx \\
\text{s.t.}\ & Ax \le D\sigma, \\
& x \ge 0.
\end{aligned} \tag{4}$$

The simplex tableau of Model (4) can be represented by:

A	I	$D\sigma$
$-\lambda C$	0	0

Given a basis B with index set J for the basic variables, let J' be the index set of the corresponding nonbasic variables. The simplex tableau of basis J can be represented as:

$B^{-1}A$	B^{-1}	$B^{-1}D\sigma$
$\lambda C_B B^{-1}A - \lambda C$	$\lambda C_B B^{-1}$	$\lambda C_B B^{-1}D\sigma$

where, C_B is the submatrix of criteria matrix corresponding to basic variables in J. Dropping the λ and σ in the simplex tableau, the MC^2 simplex tableau of basis J becomes:

$B^{-1}A$	B^{-1}	$B^{-1}D$
$C_B B^{-1}A - C$	$C_B B^{-1}$	$C_B B^{-1}D$

Set $Y(J) = [B^{-1}A, B^{-1}]$, $W(J) = [B^{-1}D]$, $Z(J) = [C_B B^{-1}A - C, C_B B^{-1}]$ and $V(J) = [C_B B^{-1}D]$. The above simplex tableau can be simplified as:

$Y(J)$	$W(J)$
$Z(J)$	$V(J)$

Let $W(J)$ and $Z(J)$ be the sub-matrix of MC^2 simplex tableau. Define

$$\Gamma(J) = \{\sigma > 0 | W(J)\sigma \geq 0\}, \tag{5}$$

$$\Lambda(J) = \{\lambda > 0 | \lambda Z(J) \geq 0\}. \tag{6}$$

The following is well known; see Yu[10] or Shi[12] for instance. (For extensive discussion of MC^2-simplex method relative to fuzzy programming, the reader is referred to Refs. 15–18).

Theorem 2.1. *The basis with index set J is a potential solution if and only if $\Lambda(J) \times \Gamma(J) \neq \emptyset$. That is, J is a potential solution if and only if there exist some $\lambda > 0$ and $\sigma > 0$ such that J is the index set of the optimal basic variables for problem* (4).

We can generate all potential bases or solutions together with their parameter spaces Γ and Λ. An example is provided Sec. 3.

3. Parameter Changes Through Investment on Resources and Marketing

It is well known that companies can change the marketing condition through investment in advertisement, service and distribution channels, etc. We shall aggregate the impacts on the changes of marketing conditions by the change of the coefficient, c, of the objective function. Likewise, we aggregate the impact of the investment effort (such as making alliance, outsourcing, extra resource allocation ...) on the resource availability by the change of d, the level of resource availability.

In order to facilitate the presentation, we will start with a concrete illustration of a simple example in Sec. 3.1. Then the concepts are then generalized in Sec. 3.2.

3.1. *A concrete illustration*

Example 3.1. A company produces two types of products, denoted by Type I and Type II, using two kinds of resources said material resource and human resource. The available resource levels of material and human resource are 100 and 120 units, respectively. Unit profits, resource consumption rates and available resource levels are summarized in Table 1. Note that unit profits of Type I and Type II products can make -3 and -5 units of profits, respectively. In other words, producing Type I or/and Type II products will not make profits at current setting. The decision maker wishes to find the optimal products mix for the company by the mathematical programming model.

According to Table 1, we can set the linear programming Model (7) as follows:

$$\begin{aligned} \max\ & -3x_1 - 5x_2 \\ \text{s.t. } & 5x_1 + 3.5x_2 \leq 100, \\ & 2.5x_1 + 2x_2 \leq 120, \\ & x_1, x_2 \geq 0, \end{aligned} \tag{7}$$

where x_1 and x_2 are decision variables representing the production units of Type I and Type II products, respectively.

The optimal solution for Model (7) is $(x_1, x_2) = (0, 0)$ and the objective value of the model is 0. In other words, since no profit can be made, the optimal decision is not producing any products.

Assume that per unit of investment, the profit rates of Type I and Type II products can be improved by 0.4 and 0.3 units, respectively, and the available resource levels of material resource and human resource can be improved by 2.5 and 1 units, respectively, as shown in Table 2.

Table 1. Unit profits, resource consumption rates and available resource levels in Example 1.

Resource	Type I	Type II	Available Resource Level
Material resource	5	3.5	100
Human resource	2.5	2	120
Unit profits of products	-3	-5	

Table 2. Profit rates and resource levels change for each unit of investment.

Resource	Type I	Type II	Available Resource Level	Change Rates for Resource Level by Investment
Material resource	5	3.5	100	2.5
Human resource	2.5	2	120	1
Unit profits of products	−3	−5		
Change rates for unit profit in time	0.4	0.3		

Let y and z be the investment put into improving the profit rate and resource availability respectively. Assume there are upper limit constraints: $y \le 200$, $z \le 300$, and $y + z \le 400$.

With this new information, we can formulate Model (8) to solve the problem.

$$\begin{aligned} \max\ & (-3x_1 - 5x_2) + y(0.4x_1 + 0.3x_2) \\ \text{s.t.}\ & 5x_1 + 3.5x_2 \le 100 + 2.5z, \\ & 2.5x_1 + 2x_2 \le 120 + z, \\ & 0 \le y \le 200, \\ & 0 \le z \le 300, \\ & y + z \le 400, \\ & x_1, x_2 \ge 0. \end{aligned} \tag{8}$$

Model (8) is a mathematical programming problem. Since, we are more interested in the impact of the solution changes for the changes of the parameter of y and z, we formulate Model (8) into Model (9) in matrix form, with extra constraint (10). The constraints of (10) are assumed to be imposed on the investment in the market and the resources, including upper limits and the total investment.

$$\begin{aligned} \max\quad & \begin{bmatrix} 1 & y \end{bmatrix} \begin{bmatrix} -3 & -5 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \text{s.t.}\quad & \begin{bmatrix} 5 & 3.5 \\ 2.5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 100 & 2.5 \\ 120 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix}, \\ & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ge 0. \end{aligned} \tag{9}$$

$$0 \le \begin{bmatrix} y \\ z \\ y + z \end{bmatrix} \le \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}. \tag{10}$$

Note that Model (9), excluding (10), is a basic MC^2-simplex format. Using MC^2-simplex method, we can locate all potential solutions (or bases) and their corresponding MC^2-simplex tableaus as listed in Table 3.

Table 3. MC^2-simplex tableaus for the potential bases of Model (9).

	x_1	x_2	x_3	x_4	RHS	
			Basis of x_3 and x_4; $J = \{3, 4\}$			
x_3	5.0000	3.5000	1.0000	0.0000	100.0000	2.5000
x_4	2.5000	2.0000	0.0000	1.0000	120.0000	1.0000
	3.0000	5.0000	0.0000	0.0000	0.0000	0.0000
	−0.4000	−0.3000	0.0000	0.0000	0.0000	0.0000
			Basis of x_1 and x_4; $J = \{1, 4\}$			
x_1	1.0000	0.7000	0.2000	0.0000	20.0000	0.5000
x_4	0.0000	0.2500	−0.5000	1.0000	70.0000	−0.2500
	0.0000	2.9000	−0.6000	0.0000	−60.0000	−1.5000
	0.0000	−0.0200	0.0800	0.0000	8.0000	0.2000
			Basis of x_1 and x_2; $J = \{1, 2\}$			
x_1	1.0000	0.0000	1.6000	−2.8000	−176.0000	1.2000
x_2	0.0000	1.0000	−2.0000	4.0000	280.0000	−1.0000
	0.0000	0.0000	5.2000	−11.6000	−872.0000	1.4000
	0.0000	0.0000	0.0400	0.0800	13.6000	0.1800
			Basis of x_4 and x_2; $J = \{2, 4\}$			
x_4	−0.3571	0.0000	−0.5714	1.0000	62.8571	−0.4286
x_2	1.4286	1.0000	0.2857	0.0000	28.5714	0.7143
	−4.1429	0.0000	−1.4286	0.0000	−142.8571	−3.5714
	0.0286	0.0000	0.0857	0.0000	8.5714	0.2143
			Basis of x_1 and x_3; $J = \{1, 3\}$			
x_1	1.0000	0.8000	0.0000	0.4000	48.0000	0.4000
x_3	0.0000	−0.5000	1.0000	−2.0000	−140.0000	0.5000
	0.0000	2.6000	0.0000	−1.2000	−144.0000	−1.2000
	0.0000	0.0200	0.0000	0.1600	19.2000	0.1600

The optimal parameter spaces Γ and Λ for each potential basis can be computed systematically using Table 3. Note that $\lambda = (1, y)$ and $\sigma = (1, z)$. Take the basis of x_3 and x_4, i.e. $J = \{3, 4\}$, as an example. By solving:

$$100 + 2.5z \geq 0, \quad 120 + z \geq 0, \quad z \geq 0,$$

we have the optimal range of z value is: $z \geq 0$. Thus, $\Gamma(\{3, 4\}) = \{z|z \geq 0\}$. Similarly, by solving:

$$3 - 0.4y \geq 0, \quad 5 - 0.3y \geq 0, \quad y \geq 0,$$

we have the optimal range of y is: $0 \leq y \leq 7.5$. Thus, $\Lambda(\{3, 4\}) = \{y|0 \leq y \leq 7.5\}$.

The optimal parameter spaces in terms of y and z for other potential solutions can be computed similarly. We list the results in Table 4.

Table 4 offers the information of the potential solution structure for Model (9), including each potential solution/basis J together with its $\Lambda(J) \times \Gamma(J)$ in terms of

Table 4. The optimal parameter spaces for the potential solutions of Model (9).

	y		z	
Basis	Lower Bound	Upper Bound	Lower Bound	Upper Bound
$J = \{3, 4\}$	0	7.5	0	Infinite
$J = \{1, 4\}$	7.5	145	0	280
$J = \{1, 2\}$	145	Infinite	146.67	280
$J = \{2, 4\}$	14.2857	Infinite	0	86.3158
$J = \{1, 3\}$	6.6667	15.8621	120	Infinite

y and z. This information can be depicted as in Fig. 1. Note in Fig. 1 by setting $\Theta(J) = \Lambda(J) \times \Gamma(J)$, we see that J is the optimal basis when $(y, z) \in \Theta(J)$.

Now, let us consider the investment constraint (10), which can be depicted as shown in Fig. 2.

Calculating the corner points of the feasible parameter space in Fig. 2, we obtain:

(i) When $(y, z) = (200, 0)$, optimal $(x_1, x_2) = (0, 28.57)$, and the optimal objective value is 1571.43.
(ii) When $(y, z) = (200, 200)$, optimal $(x_1, x_2) = (64, 80)$, and the optimal objective value is 9328.
(iii) When $(y, z) = (100, 300)$, optimal $(x_1, x_2) = (168, 0)$, and the optimal objective value is 6216.
(iv) When $(y, z) = (0, 300)$, optimal $(x_1, x_2) = (0, 0)$, and the optimal objective value is 0.

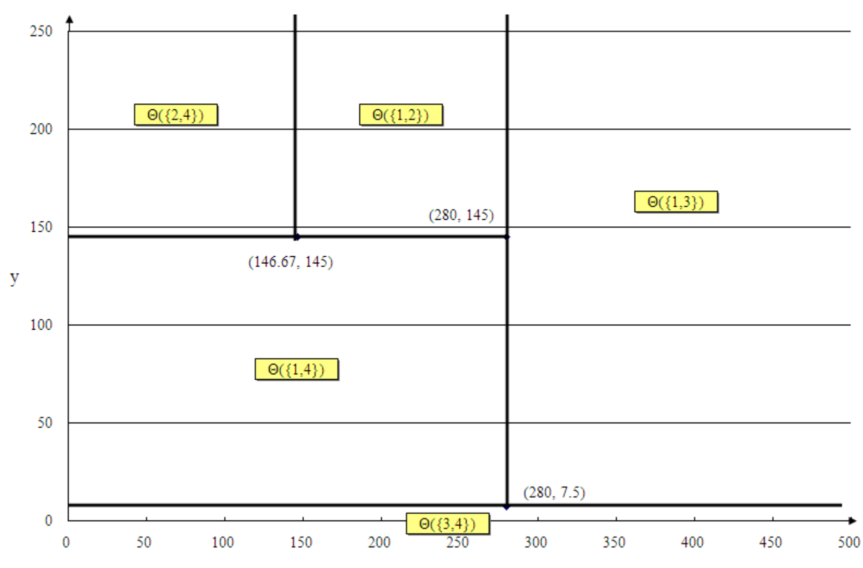

Fig. 1. The potential solution structure of Model (9) when y and z are not limited.

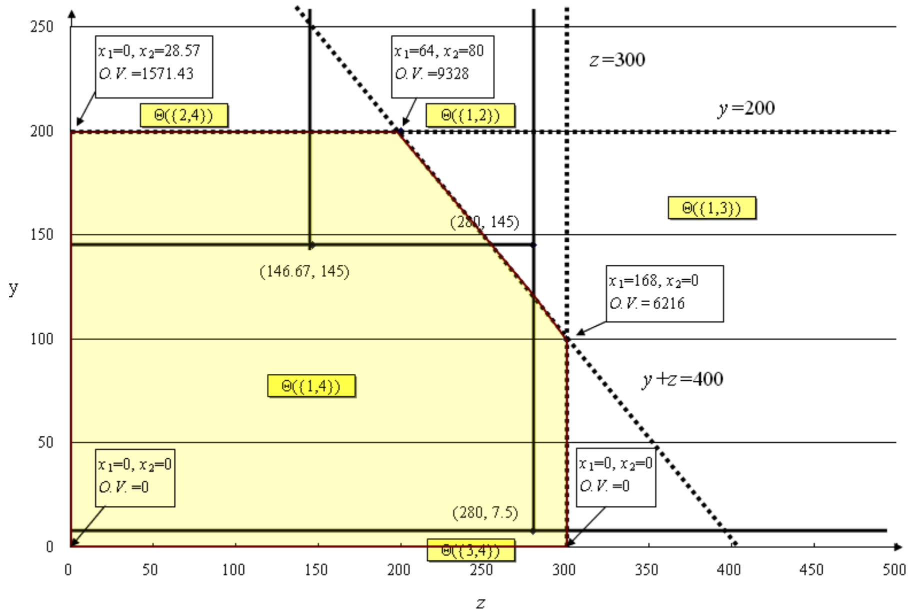

Fig. 2. The potential solution structure of Model (9) with investment constraints (10).

Figure 2 offers useful information. By varying the constraints on (y, z), the decision maker can have a full spectrum of decision outcomes, which can lead to his/her final decision on (y, z) and (x_1, x_2).

We note that in the original Model (7), the optimal solution is $(x_1, x_2) = (0, 0)$ with objective value equal to 0. By the change of the parameter (y, z), the optimal solution changes with larger objective value. This change and improvement are due to the change of relevant parameters, which is an important method to expand our habitual domains as to improve our life. For the details of this method and others see Refs. 2 and 3.

Let us generalize the above concrete illustration in the following subsection.

3.2. *Parameter changes by investment in resources and markets*

Assume that objective coefficients, namely, elements of vector c, of Model (1) are linear functions of the capital investment, which can be represented by Eq. (11).

$$c_j = f_{c_j}(y) = c_{j,0} + c_{j,1}y, \quad j = 1, \ldots, n, \tag{11}$$

where $c_{j,0}$ is the original profit rate, $c_{j,1}$ is the increased profit rate for each investment unit, and y is the investment for increasing unit profit rates.

Assume that the available resource levels, namely, elements of array d, are linear functions of the capital investment, which can be represented by Eq. (12).

$$d_i = f_{d_i}(z) = d_{i,0} + d_{i,1}z, \quad i = 1, \ldots, m, \tag{12}$$

where $d_{i,0}$ is the original available resource level, $d_{i,1}$ is the increased unit for each investment unit, and z is the investment for increasing resource available levels.

By the definition of c and d in Eqs. (11) and (12), Model (1) can be generalized as:

$$\begin{aligned} \max\ & (c_{1,0}+c_{1,1}y)x_1+\cdots+(c_{j,0}+c_{j,1}y)x_j+\cdots+(c_{n,0}+c_{n,1}y)x_n \\ \text{s.t.}\ & a_{11}x_1+\cdots+a_{1j}x_j+\cdots+a_{1n}x_n \le (d_{1,0}+d_{1,1}z), \\ & \vdots \\ & a_{i1}x_1+\cdots+a_{ij}x_j+\cdots+a_{in}x_n \le (d_{i,0}+d_{i,1}z), \\ & \vdots \\ & a_{m1}x_1+\cdots+a_{mj}x_j+\cdots+a_{mn}x_n \le (d_{m,0}+d_{m,1}z), \\ & x_j \ge 0, \quad j=1,2,\ldots,n, \\ & y,z \ge 0. \end{aligned} \tag{13}$$

Model (13) in matrix form can be represented as Problem (14).

$$\begin{aligned} \max\ & \lambda Cx \\ \text{s.t.}\ & Ax \le D\sigma, \\ & x \ge 0, \\ & \lambda \ge 0, \\ & \sigma \ge 0, \end{aligned} \tag{14}$$

where $\lambda = (1,y)$, $\sigma = (1,z)$, $x = [x_1x_2\cdots x_n]^{\mathrm{T}}$ is decision vector, $C = \begin{bmatrix} c_{1,0} & c_{2,0} & \cdots & c_{n,0} \\ c_{1,1} & c_{2,1} & \cdots & c_{n,1} \end{bmatrix}$ is the profit rates matrix (including original objective coefficients and the change rates of profit by investment), $A = [a_{ij}]$ $(i = 1,\ldots,m;$ $j = 1,\ldots,n)$ is the resource consumption matrix, and $D = \begin{bmatrix} d_{1,0} & d_{2,0} & \cdots & d_{m,0} \\ d_{1,1} & d_{2,1} & \cdots & d_{m,1} \end{bmatrix}^{\mathrm{T}}$ is the available resource levels matrix (including original available resource levels and increased units of resource by each unit of investment).

Model (14) can be solved by MC^2-simplex method. The set of all potential solutions/bases can be obtained systematically as illustrated in Sec. 3.1.

Observe that in Problem (14), there are only two parameters, y and z, that are subject to change. Useful information provided by Table 4 and Fig. 1 of Example 3.1 can also be constructed for Problem (14).

Suppose that there are constraints imposed on the investment. We can easily add them to Problem (14), like inequality (10) adding to Problem (9). We shall not stop to do so.

Note that the constraints on investment offer useful information for final decision. Nevertheless, the constraints itself can also be subject to change. A bright decision maker certainly would like to keep this option for better decision.

4. "Red in-Black Out" Phenomenon-Parameter Changes in c and d due to Time Advancement

In this section, we will focus on the parameter changes of c and d due to time advancement. That is, c and d are both functions of time, or $c = c(t)$ and $d = d(t)$. We shall start with a concrete simple example in Sec. 4.1. Then generalize the concept in Sec. 4.2.

4.1. *An illustrative example*

Example 4.1. (Continue on Example 3.1) Assume the original unit profits, resource consumption rate and available resource levels are same as Table 1 with linear programming model as Model (7). However, for each unit of time advancement, the unit profit of Type I and Type II products will increase by 0.4 and 0.3 unit respectively; and the resource available level for material and human resource will increase by 2.5 and 1 unit, respectively. The problem can be summarized as in Table 5.

The problem can be formulated as in Model (15).

$$\begin{aligned} \max\ & (-3x_1 - 5x_2) + t(0.4x_1 + 0.3x_2) \\ \text{s.t.}\ & 5x_1 + 3.5x_2 \leq 100 + 2.5t, \\ & 2.5x_1 + 2x_2 \leq 120 + t, \\ & x_1, x_2, t \geq 0. \end{aligned} \tag{15}$$

Note that Model (15) has only one parameter, t, while Model (8) of Example 3.1 has two parameters y and z. By setting t at different values, we can obtain the corresponding optimal solutions and the objective values as summarized in Table 6.

The useful information of Table 6 can be further depicted as shown in Fig. 3.

In Fig. 3, when $0 \leq t < 7.5$, the optimal basis is $J = \{3, 4\}$ and the optimal objective value is 0, the decision of not producing any product is made due to the fact that no profit can be made. When $7.5 \leq t < 145$, the optimal basis is $J = \{1, 4\}$ and the optimal objective value increases by time and only Type I product is produced. When $145 \leq t < 146.67$, the optimal basis is $J = \{2, 4\}$ and the optimal objective value increases by time and only Type II product is produced.

Table 5. Example 4.1 in a nut shell.

Resource	Type I	Type II	Available Resource Level	Change Rates for Resource Level by Investment
Material resource	5	3.5	100	2.5
Human resource	2.5	2	120	1
Unit profits of products	-3	-5		
Change rates for unit profit in time	0.4	0.3		

Table 6. The optimal solutions and their objective values for different t values for Problem (15).

t	0	7.5	7.6	50	100	144	146
x_1	0	0	23.8	45	70	92	0
x_2	0	0	0	0	0	0	132.86
Optimal basis	$J(3,4)$			$J(1,4)$			$J(2,4)$
Objective value	0	0	0.95	765	2590	5023.2	5154.86
t	150	200	250	279	280	300	
x_1	4	64	124	158	160	168	
x_2	130	80	30	1	0	0	
Optimal basis		$J(1,2)$				$J(1,3)$	
Objective value	5428	9328	14128	17324.38	17440	19656	

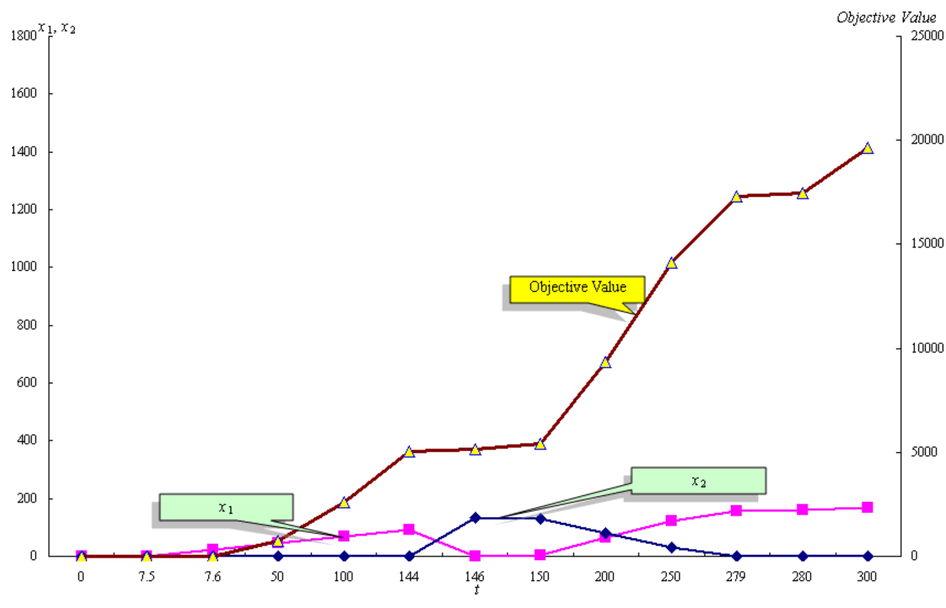

Fig. 3. Trends of optimal solutions and objective values at different time interval.

When $146.67 \leq t < 280$, the optimal basis is $J = \{1, 2\}$ and the optimal objective value increases in acceleration by time and both Type I and Type II products are produced. When $t \geq 280$, the optimal basis is $J = \{1, 3\}$ and the optimal objective value increases in time and only Type I product is produced. Note that in Fig. 3, the critical times of transition are monotonically, not proportionally, deployed.

Note that the changing pace of time for parameters in the objective function, i.e. array c, and in the constraints availability, i.e. array d, is the same. Therefore, the critical time for the changes of optimal bases in Fig. 3 can be calculated by depicting a line, $y = z$, in Fig. 1, which is shown in Fig. 4. The intersecting points

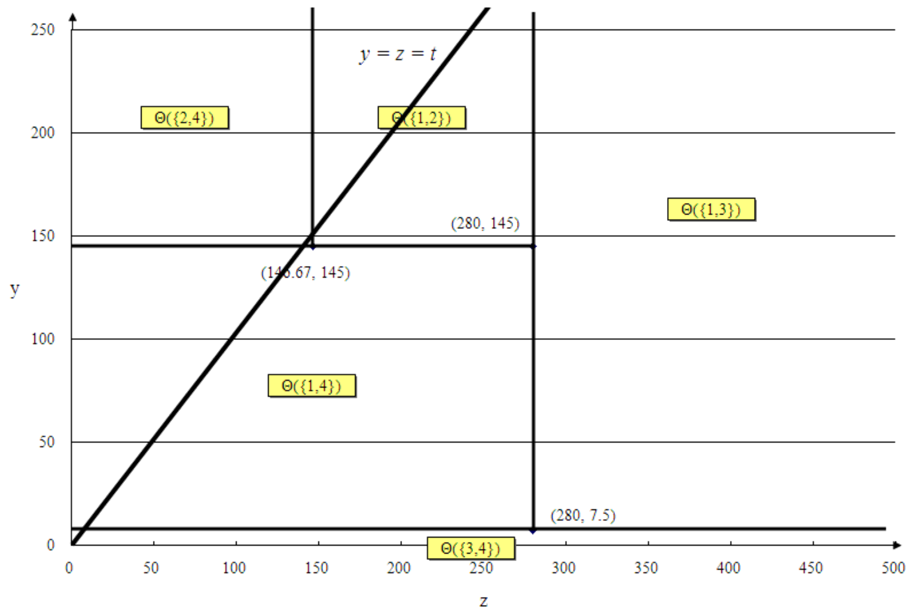

Fig. 4. Intersecting points of the line of $y = z$ with the potential solution structure in parameter space.

of the line $y = z$ and the range of the different optimal bases are corresponding to the critical time points shown in Table 6 and Fig. 3.

Finally observe, from Table 6 and Fig. 3, because of that optimization formulation, whenever the optimal objective value is zero, the products should not be produced due to the fact that each product produced will bring negative profit or deficit (red in). However, when $t \geq 7.5$, Type I product began to be able to bring in positive profit (black out). If the delivery time is set at some time $t > 7.5$, then positive profit can be fulfilled. We shall further discuss this subject in the following subsection.

4.2. *Generalized model for parameter changes in c and d due to time advancement*

With concrete Example 4.1 in mind, assume that $c(t)$ and $d(t)$ are linear. More specifically, let

$$c_j(t) = c_j = c_{j,0} + c_{j,2}t, \quad j = 1, \ldots, n, \tag{16}$$

where $c_{j,0}$ is the original profit rate, $c_{j,2}$ is the increased profit for each unit of time passed, and

$$d_i(t) = d_i = d_{i,0} + d_{i,2}t, \quad i = 1, \ldots, m, \tag{17}$$

where $d_{i,0}$ is the original available resource level, $d_{i,2}$ is change rate of resource availability over time.

Introducing Eqs. (16) and (17) into Model (1), we obtain the following changeable parameter model over c and d due to time advancement.

$$\begin{aligned}
\max\ & (c_{1,0}+c_{1,2}t)x_1+(c_{2,0}+c_{2,2}t)x_2+\cdots+(c_{n,0}+c_{n,2}t)x_n \\
\text{s.t.}\ & a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n \le (d_{1,0}+d_{1,2}t), \\
& a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n \le (d_{2,0}+d_{2,2}t), \\
& \vdots \\
& a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n \le (d_{m,0}+d_{m,2}t), \\
& x_j \ge 0, \quad j=1,2,\ldots,n, \\
& t \ge 0.
\end{aligned} \tag{18}$$

Note that Problem (18) has only one parameter t. It can be formulated as

$$\begin{aligned}
\max\ & \begin{bmatrix}1 & t\end{bmatrix}\begin{bmatrix} c_{1,0} & c_{2,0} & \cdots & c_{n,0} \\ c_{1,2} & c_{2,2} & \cdots & c_{n,2}\end{bmatrix} x \\
\text{s.t.}\ & Ax \le \begin{bmatrix} d_{1,0} & d_{1,2} \\ d_{2,0} & d_{2,2} \\ \cdots & \cdots \\ d_{m,0} & d_{m,2}\end{bmatrix}\begin{bmatrix}1 \\ t\end{bmatrix}, \\
& x_j \ge 0, \quad j=1,2,\ldots,n, \\
& t \ge 0.
\end{aligned} \tag{19}$$

By varying t, for $t \ge 0$, one can generate the useful information such as those of Table 6 and Fig. 3 for final decision.

The phenomenon of "Red in-Black out" can be roughly explained as: at the ordering time ($t = 0$), the optimal objective value is less than or equal to 0, and at the delivery time ($t = t_1 > 0$), the optimal objective value is greater than 0 because the parameters have been changed over time. The following results can help the company to figure out if "Red in-Black out" is a good strategic decision or not.

Given j, define $I(j) = \{i | a_{ij} > 0\}$.

Proposition 4.1. *Assume there exist $j \in \{1,\ldots,n\}$ such that for all $i \in I(j)$, $c_{j,2} > 0$ and $d_{i,2} > 0$. Then, as time advances, Problem* (18) *will eventually make profit.*

Proof. Set

$$t_j = \min_t \{t | c_{j,0} + tc_{j,2} \ge 0 \quad \text{and} \quad d_{i,0} + td_{i,2} \ge 0, \qquad \text{for all } i \in I(j)\}. \tag{20}$$

Then, for all $i \in I(j)$, when $t > t_j$, $c_{j,0} + tc_{j,2} > 0$ and $d_{i,0} + td_{i,2} > 0$, the production solution x^* (with $x_j^* = \min_t\{\frac{d_{i,0}+td_{i,2}}{a_{ij}}\} > 0$, and $x_k^* = 0$, for all

$k \neq j$) will make a positive profit because of $c_{j,0} + tc_{j,2} > 0$, the objective function, $(c_{j,0} + tc_{j,2})x_j^* > 0$, which is smaller than the optimal objective value of Problem (18). □

Remark 4.1. Suppose there exist j, such that $c_{j,0} > 0$ and for all $i \in I(j)$, $d_{i,0} > 0$. Then according to Eq. (20) of the above proof, the production system will make profit at time 0.

For a given j and small $\varepsilon > 0$, define

$$s_j(\varepsilon) = \min_s \{c_{j,0} + sc_{j,2} \geq 0 \text{ and } d_{i,0} + sd_{i,2} \geq \varepsilon, \text{ for all } i \in I(j), \ s \geq 0\}. \tag{21}$$

Note that s_j can be an empty set, or not defined if $c_{j,0} + sc_{j,2}$ and $d_{i,0} + sd_{i,2}$, $i \in I(j)$ cannot be greater than 0 at the same time such as those shown in Fig. 5.

Proposition 4.2. (i) *Suppose there exist j, such $s_j(\varepsilon)$ is not an empty set, then $s_j(\varepsilon)$ is the time point at which the Problem* (18) *will not yield loss. Furthermore, if $c_{j,0} + s_j(\varepsilon)c_{j,2} > 0$, then $s_j(\varepsilon)$ is a time point at which the Problem* (18) *will yield profit.*
(ii) *For all j such that $c_{j,0} < 0$, let $s^*(\varepsilon) = \min_j\{s_j(\varepsilon)\}$. Then for time $t > s^*(\varepsilon)$, the system of* (18) *can make profit.*

Proof. For (i). It can be proved similar to that of Proposition 4.1.

For (ii). Suppose $s^*(\varepsilon) = s_k(\varepsilon)$. As $c_{k,0} < 0$, $c_{k,0} + s_k(\varepsilon)c_{k,2} \geq 0$, $c_{k,0} + tc_{k,2} > 0$ for $t > s_k(\varepsilon)$. Therefore, the system can make a profit when $t > s_k(\varepsilon) = s^*(\varepsilon)$. □

Assume the production system will make profit eventually, which can be checked by above Propositions 4.1 and 4.2, and that the optimal objective function value $v(t)$ is increasing with time with $v(t = 0) \leq 0$. Let t_0 be *the earliest critical time* of making profit in the sense that when $t > t_0$ the system can make profit with $v(t) > 0$; and when $t < t_0$ the system will not make profit with $v(t) \leq 0$. The following algorithm, exemplified by the flow chart of Fig. 6, can be of help to find t_0.

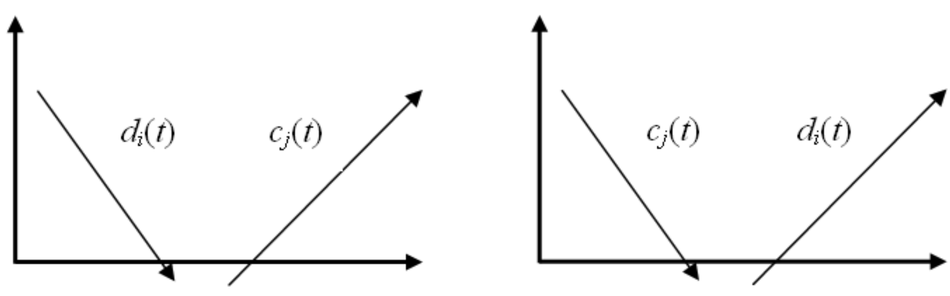

Fig. 5. Two situations for s_j to be an empty set.

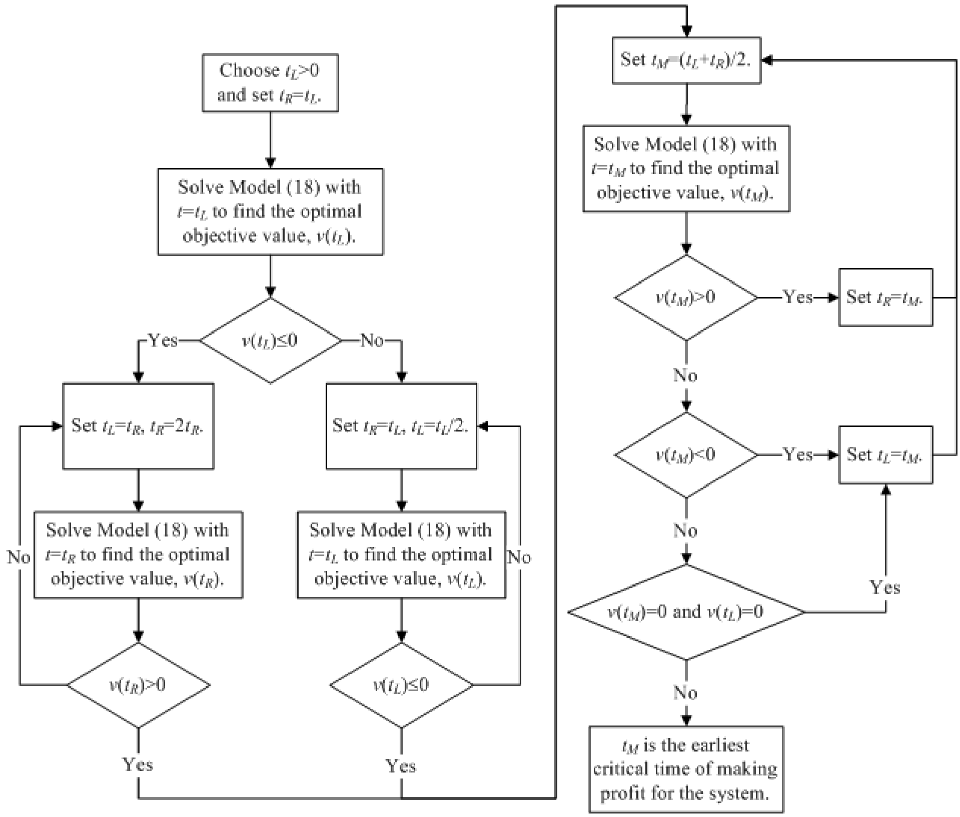

Fig. 6. Flow chart of Algorithm 4.1.

Algorithm 4.1.

Step 1. Choose $t_L > 0$, where t_L denotes *left end point*, and set $t_R = t_L$, where t_R denotes *right end point*.

Step 2. Solve Model (18) with $t = t_L$ to obtain the optimal objective value, $v(t_L)$.

Step 3. If $v(t_L) \le 0$, go to Step 3.1–3.3. Otherwise, go to Step 4.

- **Step 3.1.** Set $t_L = t_R$, $t_R = 2t_R$.
- **Step 3.2.** Solve Model (18) with $t = t_R$ to obtain the optimal objective value, $v(t_R)$.
- **Step 3.3.** If $v(t_R) > 0$, go to Step 5. Otherwise, back to Step 3.1.

Step 4. If $v(t_L) > 0$, go to Steps 4.1–4.3.

- **Step 4.1.** Set $t_R = t_L$, $t_L = t_L/2$.
- **Step 4.2.** Solve Model (18) with $t = t_L$ to obtain the optimal objective value, $v(t_L)$.
- **Step 4.3.** If $v(t_L) \le 0$, go to Step 5. Otherwise, back to Step 4.1.

Step 5. Set $t_M = (t_L + t_R)/2$, where t_M denotes the *middle point* of the interval $[t_L, t_R]$.

Step 6. Solve Model (18) with $t = t_M$ to find the optimal objective value, $v(t_M)$.

Step 6.1. If $v(t_M) > 0$, set $t_R = t_M$ and back to Step 5.

Step 6.2. If $v(t_M) < 0$, set $t_L = t_M$ and back to Step 5.

Step 6.3. If $v(t_M) = 0$ and $v(t_L) = 0$, set $t_L = t_M$ and back to Step 5; if $v(t_M) = 0$ and $v(t_L) < 0$, then the time point t_M is the earliest critical time of making profit for the system.

Theorem 4.1. *If system* (18) *will make profit eventually and the objective function value will increase by time*, *Algorithm* 4.1 *will converge.*

Proof. The proof of this theorem is similar to that of bisection method. For details, see Ref. 19. □

5. Generalized Model for Parameter Changes Including Elements of *A*

It is well known that parameter changes in elements of A usually involve nonlinear computation for optimization. However, when the changes follow some specific pattern, the mathematical programming can be reduced to a form of linear inequalities with multi-level resource availability constraints.

Again, we will start with a concrete simple example in Sec. 5.1. The generalized model for parameter changes including elements of A and its relation to "Red in-Black out" phenomenon will be given in Sec. 5.2. Further generalization will be given in Sec. 5.3.

5.1. *An illustrative example*

Example 5.1. (Continue on Example 4.1) Assume that the consumption of resources, perhaps due to technological advancement, is reduced at a rate $(1 + 0.025t)^{-1}$ and $(1 + 0.00833t)^{-1}$, respectively for material and human resource and the resource availability remains the same. Table 7 offers a summary of the problem.

Table 7. A summary of the problem of Example 5.1.

Resource	Type I	Type II	Available Resource Level	Change Rates for Resource Level by Investment
Material resource	5	3.5	100	$(1 + 0.025t)^{-1}$
Human resource	2.5	2	120	$(1 + 0.00833t)^{-1}$
Unit profits of products	−3	−5		
Change rates for unit profit in time	0.4	0.3		

Note that the objective function is the same as in Problem (15). The constraints of the problem can be rewritten as:

$$\begin{bmatrix} 5*(1+0.025t)^{-1} & 3.5*(1+0.025t)^{-1} \\ 2.5*(1+0.00833t)^{-1} & 2*(1+0.00833t)^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 100 \\ 120 \end{bmatrix} \tag{22}$$

or

$$5x_1 + 3.5x_2 \leq 100*(1+0.025t) = 100 + 2.5t$$

$$2.5x_1 + 2x_2 \leq 120*(1+0.00833t) = 120 + t$$

which reduces to

$$\begin{bmatrix} 5 & 3.5 \\ 2.5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 100 & 2.5 \\ 120 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix}. \tag{23}$$

Note the constraint (23) is identical to that of Problem (15). Therefore, all the discussion and computation for useful information of Problem (15) can be carried over to this new problem. We shall not repeat it. Being limited by space, we purposefully choose the change rates for resource usage in time so that we do not have to repeat the computation. Of course, the model can be applied to different rate of change in resource usage.

5.2. *A generalization, including changes in elements of A*

Assume that elements of c, d, and A can be changed over time and they are all linear functions of time. Specifically,

(i) the objective coefficients, c, can be represented by Eq. (24).

$$c_j = c_{j,0} + c_{j,2}t, \quad j = 1, \ldots, n, \tag{24}$$

where $c_{j,0}$ is the original profit rate, $c_{j,2}$ is the increased profit for each unit of time, and t represents the time units.

(ii) the elements of matrix A, will be changed over time and can be represented by Eq. (25).

$$a_{ij} = \frac{a_{ij,0}}{1 + a_{ij,1}t}, \quad i = 1, \ldots, m, \; j = 1, \ldots, n, \tag{25}$$

where $a_{ij,0}$ is the original consumption rate for different product j in resource i; $a_{ij,1}$ is the change rate for each unit of time for different product j in resource i. (Note that if for each i, $a_{ij,1}$ is the same for all j, then the constraints reduces to a similar form of Eq. (23).)

(iii) the resource available level, namely, elements of d, can be represented by Eq. (26).

$$d_i = d_{i,0} + d_{i,2}t, \quad i = 1, \ldots, m, \tag{26}$$

where $d_{i,0}$ is the original available resource level and $d_{i,2}$ is change rate of available resource over time.

Introducing Eqs. (24)–(26) into Model (1), we obtain the following changeable parameter model due to time advancement shown in Model (27).

$$
\begin{aligned}
\max\ & (c_{1,0}+c_{1,2}t)x_1+(c_{2,0}+c_{2,2}t)x_2+\cdots+(c_{n,0}+c_{n,2}t)x_n\\
\text{s.t.}\ & [a_{11,0}/(1+a_{11,1}t)]x_1+\cdots+[a_{1n,0}/(1+a_{1n,1}t)]x_n\le(d_{1,0}+d_{1,2}t),\\
& [a_{21,0}/(1+a_{21,1}t)]x_1+\cdots+[a_{2n,0}/(1+a_{2n,1}t)]x_n\le(d_{2,0}+d_{2,2}t),\\
& \vdots\\
& [a_{m1,0}/(1+a_{m1,1}t)]x_1+\cdots+[a_{mn,0}/(1+a_{mn,1}t)]x_n\le(d_{m,0}+d_{m,2}t),\\
& x_j\ge 0,\quad j=1,2,\ldots,n,\\
& t\ge 0.
\end{aligned}
\tag{27}
$$

Similar to Propositions 4.1–4.2, Algorithm 4.1, and Theorem 4.1, we can restate their general cases as follows. Recall that $I(j)=\{i|a_{ij}>0\}$.

Proposition 5.1. *Assume there exist $j\in\{1,\ldots,n\}$ such that for all $i\in I(j)$, $c_{j,2}>0$ and $d_{i,2}>0$. Then, as time advances, Problem* (27) *will eventually make profit.*

Recall that $s_j(\varepsilon)=\min_s\{c_{j,0}+sc_{j,2}\ge 0$ and $d_{i,0}+sd_{i,2}\ge\varepsilon$, for all $i\in I(j)$, $s\ge 0\}$, as defined in (21).

Proposition 5.2. (i) *Suppose there exist j, such $s_j(\varepsilon)$ is not an empty set, then $s_j(\varepsilon)$ is the time point at which the Problem* (27) *will not yield loss. Furthermore, if $c_{j,0}+s_j(\varepsilon)c_{j,2}>0$, then $s_j(\varepsilon)$ is a time point at which the Problem* (27) *will yield profit.*

(ii) *For all j such that $c_{j,0}<0$, let $s^*(\varepsilon)=\min_j\{s_j(\varepsilon)\}$. Then for time $t>s^*(\varepsilon)$, the system of* (27) *can make profit.*

Algorithm 5.1.

Step 1. Choose $t_L>0$, where t_L denotes *left end point*, and set t_R=t_L, where t_R denotes *right end point*.
Step 2. Solve Model (27) with $t=t_L$ to obtain the optimal objective value, $v(t_L)$.
Step 3. If $v(t_L)\le 0$, go to Steps 3.1–3.3. Otherwise, go to Step 4.

- **Step 3.1.** Set $t_L=t_R$, $t_R=2t_R$.
- **Step 3.2.** Solve Model (27) with $t=t_R$ to obtain the optimal objective value, $v(t_R)$.
- **Step 3.3.** If $v(t_R)>0$, go to Step 5. Otherwise, back to Step 3.1.

Step 4. If $v(t_L)>0$, go to Steps 4.1–4.3.

- **Step 4.1.** Set $t_R=t_L$, $t_L=t_L/2$.
- **Step 4.2.** Solve Model (27) with $t=t_L$ to obtain the optimal objective value, $v(t_L)$.
- **Step 4.3.** If $v(t_L)\le 0$, go to Step 5. Otherwise, back to Step 4.1.

Step 5. Set $t_M = (t_L + t_R)/2$, where t_M denotes the *middle point* of the interval $[t_L, t_R]$.

Step 6. Solve Model (27) with $t = t_M$ to find the optimal objective value, $v(t_M)$.

- **Step 6.1.** If $v(t_M) > 0$, set $t_R = t_M$ and back to Step 5.
- **Step 6.2.** If $v(t_M) < 0$, set $t_L = t_M$ and back to Step 5.
- **Step 6.3.** If $v(t_M) = 0$ and $v(t_L) = 0$, set $t_L = t_M$ and back to Step 5; if $v(t_M) = 0$ and $v(t_L) < 0$, then the time point t_M is the earliest critical time of making profit for the system.

Theorem 5.1. *If the production system will make profit eventually and the objective function value is increasing with time*, *Algorithm* 5.1 *will converge.*

5.3. *Further generalization with parameters as control variables*

Let k is the investment units for changing the efficiency of resource usage. Model (13) of Sec. 3.2 can be further expanded as follows.

Let

$$a_{ij} = f_{a_{ij}}(k) = \frac{a_{ij,0}}{1 + a_{ij,1}k}, \quad i = 1, \dots, m;\ \ j = 1, \dots, n, \tag{28}$$

where $a_{ij,0}$ is the original consumption rate for different product j in resource i; $a_{ij,1}$ is the change rate for each unit of investment for different product j in resource i.

Introducing Eq. (28) into Model (13), we obtain the following model with parameters as control variables.

$$\begin{aligned}
\max\ & (c_{1,0} + c_{1,1}y)x_1 + (c_{2,0} + c_{2,1}y)x_2 + \cdots + (c_{n,0} + c_{n,1}y)x_n \\
\text{s.t.}\ & [a_{11,0}/(1 + a_{11,1}k)]x_1 + \cdots + [a_{1n,0}/(1 + a_{1n,1}k)]x_n \le (d_{1,0} + d_{1,1}z), \\
& [a_{21,0}/(1 + a_{21,1}k)]x_1 + \cdots + [a_{2n,0}/(1 + a_{2n,1}k)]x_n \le (d_{2,0} + d_{2,1}z), \\
& \vdots \\
& [a_{m1,0}/(1 + a_{m1,1}k)]x_1 + \cdots + [a_{mn,0}/(1 + a_{mn,1}k)]x_n \le (d_{m,0} + d_{m,1}z), \\
& y \le y_M, \\
& z \le z_M, \\
& k \le k_M, \\
& y + z + k \le I_M, \\
& x_j \ge 0, \quad j = 1, 2, \dots, n, \\
& y, z, k \ge 0.
\end{aligned} \tag{29}$$

Note that the above formulation, y, z and k are changeable parameters as well as control variables. When there are other constraints imposing on y, z, k, they can be easily added on. The Problem (29) is usually nonlinear. As demonstrated before (Secs. 5.1 and 5.2), with special structure, it can be reduced to MC^2-simplex format, and can be solved systematically. We shall not repeat it.

6. Conclusions

Motivated by the "Red in-Black out" phenomenon (taking loss at the ordering time and making profit at the time of delivery), we study linear programming models with changeable parameters using multi-criteria and multi-constraint level linear programming (MC^{2}LP) models. We have provided formulations, computation methods and analysis as to gain useful insight into the "Red in-Black out" phenomenon. We have also proposed an algorithm to locate the first critical time of making profit for a given system, which is an important information to those decision makers who consider adopting the "Red in-Black out" as a business strategy. At the end, we also sketch a generalized mathematical programming model with changeable parameters and control variables to study more general cases.

Many research problems are open. For instances, how to interpret the meaning of the dual problem of the proposed model? How to deal with the uncertainty and fuzziness of parameter changes due to investment or time advancement? We invite interested readers to explore these interesting and meaningful problems.

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